Recitation 6

October 1, 2015

Problems

Problem 1. The stochastic matrix is $P = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$.

The situation of the TA being fine now is represented by the probability vector $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then the probability of him being fine after one more solution is $Pv = \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}$.

If there is a 30% chance the TA is sad now, it is represented by the vector $u = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$. Since $Pu = \begin{bmatrix} 0.69 \\ 0.31 \end{bmatrix}$, the probability of the TA being fine after one more solution is 69%.
The equilibrium vector q satisfies the equation $Pq = q$. Solving the system of equations $(P - I)q = 0$, i.e. finding the null space of $P - I = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix}$ we see that $q = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$. Thus eventually TA will be spending 1/3 of his time frustrated. Not too bad.

Problem 2. Suppose we have a linear combination
$c_1(v_1 + 3v_2 - v_3) + c_2(-v_1 - 2v_2 + v_3) + c_3(v_2 + 2v_3) = 0$. We would like to know whether this combination can be non-trivial, i.e. whether we can find not all zero scalars c_1, c_2, c_3 making the equation above true. We can re-write this equation as

$$(c_1 - c_2)v_1 + (3c_1 - 2c_2 + c_3)v_2 + (-c_1 + c_2 + 2c_3)v_3 = 0$$

But we know that the vectors v_1, v_2, v_3 are linearly independent. Therefore for the equation above to be true we must have $c_1 - c_2 = 0$, $3c_1 - 2c_2 + c_3 = 0$ and $-c_1 + c_2 + 2c_3 = 0$. For which c_1, c_2, c_3 can this be? In other words, we are really asking for solutions of the homogeneous system of equations

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & -2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row reduce, see that each column is pivotal, so there are no non-trivial solutions, so $c_1 = c_2 = c_3 = 0$, and so the required vectors are linearly independent.

Problem 3. We have $A - 2I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$. This matrix has non-trivial null space (for example, because $\det(A - 2I) = 0$). So $\lambda = 2$ is an eigenvalue.

Problem 4. $\lambda = 1$ and $\lambda = 3$ are the only eigenvalues of A.
Let's find an eigenvector for $\lambda = 1$. We need to find v s.t. $Av = 1 \cdot v = v$, i.e. we are looking for null space of $A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Row reduce, find that the null space consists of vectors of the form $x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So for example, $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is an eigenvector. Any other non-zero multipel of this one would work.
For $\lambda = 3$ we can take $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 5. For a number λ to be an eigenvalue, we need the matrix

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 3 & -3 \\ 0 & 2 - \lambda & 1 \\ 0 & 0 & 3 - \lambda \end{bmatrix}$$

to have non-trivial null-space. In other words, we want to have some free variables in the corresponding homogeneous system, or equivalently we want to have some non-pivotal columns. If λ is not equal $1, 2, 3$ then every column would be pivotal. If $\lambda = 1, 2$ or 3, one of the columns would be non-pivotal. So the eigenvalues are $1, 2, 3$.
Since eigenvalues are all distinct, the corresponding eigenvectors are linearly independent.

Problem 6. Number 0 is an eigenvalue means that for some non-zero vector $v \neq 0$ one has $Av = 0 \cdot v = 0$. Thus A has non-trivial null space, and so can't be invertible.

Problem 7. The matrix A is invertible, hence non of the eigenvalues $\lambda_1, \dots, \lambda_n$ are zero. Now, if $Av = \lambda v$ and A is invertible, we can multiply $Av = \lambda v$ by A^{-1} and get $v = \lambda A^{-1} v$, and so $A^{-1} v = \frac{1}{\lambda} v$. So $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ are the eigenvalues of A^{-1}.

Problem 8. Equilibrium vector is an eigenvector corresponding to eigenvalue 1.

Problem 9.

- False
- False
- True
- False

Problem 10. Transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ reflecting with respect to the line $x = y$ preserves any vector on this line. So $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector, and since it is being preserved, i.e. $T(v) = v$, the eigenvalue is 1.
The other line that is preserved is the line $x = -y$, since it is perpendicular to $x = y$. Each vector u on the line $x = -y$ is reflected, i.e. gets mapped to $-u$. So for example $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector, with the eigenvalue $\lambda = -1$.
Transformation $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ rotating around the y-axis by the angle $\pi/2$ preserves the y-axis, so non-zero vector on this axis (for example $v = [0, 1, 0]^T$) will be eigenvectors with the eigenvalue 1.

Problem 11. Matrix A is called nilpotent if $A^n = 0$ for some number n. Give an example of a 3×3 non-zero nilpotent matrix. What are the eigenvalues of any nilpotent matrix A?
$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is non-zero nilpotent, since $A^2 = 0$. If $Av = \lambda v$ for some non-zero $v \neq 0$, then $A^2 v = A(\lambda v) = \lambda^2 v$, $A^3 v = \lambda^3 v$, et.c. Since A ius nilpotent, at some step we get $A^n = 0$, and so we will be looking at the equation $0 = \lambda^n v$. Since $v \neq 0$, then $\lambda^n = 0$, and so $\lambda = 0$. Thus the only eigenvalue nilpotent matrix can have is 0.